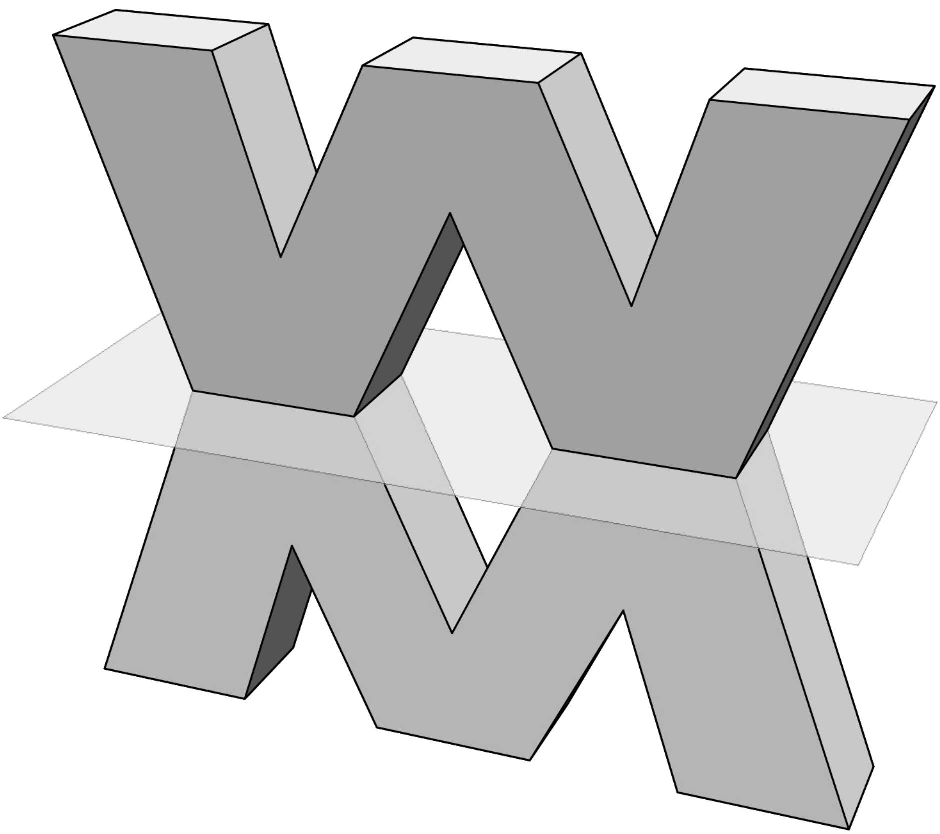
Przemysław Koprowski

Witt Morphisms



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Contents

Foreword

A bilinear form defines an orthogonal geometry on a given linear space or projective module. Once we omit degenerate forms (roughly speaking, these are forms allowing a non-zero vector to be orthogonal to the entire space), the remaining ones may be classified by the relation of similarity (the term is explained in Section 1.1). This leads us to the notion of the Witt ring, which is an algebraic structure consisting of all the similarity classes of finitely generated projective modules over a given base ring. In a sense, Witt ring describes all the possible orthogonal geometries over the ring/filed in question. The leading theme of this book is to study morphisms between Witt rings. In geometric terms, this can be viewed as analyzing to what extent the orthogonal geometries defined over one ring may be transcribed to another ring. For example, knowing the criteria for existence of an isomorphism between Witt rings, one may verify whether two rings/fields admit the same set of orthogonal geometries. If this is the case we say that the two rings are *Witt equivalent*. This problem has been intensively researched in previous years. The complete criteria are known for fields with small square class groups (see [9]) and global fields (see e.g. [44, 53, 52]). The author of this book found criteria for Witt equivalence of function fields and rings of regular function on real algebraic curves (summary of these results can be found in the appendix to Chapter 5). The criterion of Witt equivalence of real function fields has been recently generalized by N. Grenier-Boley and D. Hoffmann to arbitrary real fields with u-invariants not exceeding 2. We present their result in Section 5.1. In the second section of the last chapter we apply the ideas used earlier for rings of regular function on real curves to extend the result of Grenier-Boley and Hoffmann and obtain a necessary condition (see Theorem 5.15) for Witt equivalence of real holomorphy rings. This generalizes our earlier result obtained in [27].

Chapter 4 copes with a splitting of the Knebusch-Milnor exact sequence. A classical theorem due to M. Knebusch and J. Milnor (see Theorem 1.40) assert that the Witt ring WR of a Dedekind domain R injects into the Witt ring of its field of fractions K and the image of this embedding coincides with the kernel of a map from the Witt ring of the field to the co-product of Witt rings of all the localizations of the base ring. In a nutshell, this says that the structure of orthogonal geometries over K is at least as rich as the structure over the underlying Dedekind domain. Surprisingly, not much is known about the splitting of the above-mentioned injection $WR \rightarrowtail WK$. The problem was solved in the case of algebraic integers by P. Shastri in [50] and for real geometric rings by the author in [26] and [28]. Chapter 4 summarizes the results of these two papers. The main result of this chapter (see Theorem 4.3) asserts that, if R is the ring of regular functions on a smooth real curve, then WR is a direct summand of the Witt ring of the field of fractions of R (the field of rational functions on this curve). Consequently, the Knebusch-Milnor exact sequence slices into and is patched by two split exact sequences (c.f. Theorem 4.17). Moreover, if the curve in question is semi-algebraically compact and semi-algebraically connected, then the Witt ring of the ring of polynomial function is in turn a direct summand of WR as shown in Theorem 4.19.

On the other hand, it is natural to reckon that starting from a ring with a complex Witt ring (i.e. with a rich structure of orthogonal geometries) and appending roots of (quadratic) polynomials one can successively kill elements of the Witt ring. Therefore, it is expected that the natural morphism of Witt rings corresponding to an algebraic (resp.: real, quadratic or integral) closure of a field/ring should *not* be injective. A classical example: start from the field $\mathbb{Q}$ of rationals, let R denote its real closure and $R[\sqrt{-1}]$ be the algebraic closure. The Witt ring of $\mathbb{Q}$ has quite a complex structure (as additive group it is a direct sum of infinitely many nontrivial term, see [33, Chapter VI, Section 4]), while WR is isomorphic to the ring of integers and $WR[\sqrt{-1}]$ consists of just two elements. Thus, the injections $\mathbb{Q} \rightarrowtail R \rightarrowtail R[\sqrt{-1}]$ correspond to the maps $W\mathbb{Q} \to WR \to WR[\sqrt{-1}]$, both having strongly nontrivial kernels. In Chapter 3 we concentrate on an analogy of this phenomenon in the case of the integral closure of a ring. For example, we show (see Theorem 3.15) that if P is seminormal but not quadratically closed and R is the integral closure of P, then the natural morphism $WP \to WR$ is not injective. This problem has also a natural interpretation in terms of the Picard functor. This connection is studied in Section 3.3. We close this chapter showing how to apply these results in the case of curve desingularization.

The Witt functor of a ring extension is also the subject of the second chapter. Here, however, we consider a quadratic extension of a local ring. We develop a generalization of Scharlau's transfer and prove an analogy of Scharlau's norm principle. This allows us to construct examples of ring extensions where both rings have the same field of fractions but the corresponding Witt morphism is not surjective (hence there are classes of forms over the bigger ring not comming from the smaller one).

The new contributions in this book include: entire Chapters 2 and 3 and Section 5.2 where the main new results are Theorems: 2.17, 2.19, 2.24, 3.11, 3.15, 3.24, 5.15 and Proposition 2.35. The results of Chapter 4 appeared earlier in

[26, 28]. The proofs presented here are only slightly improved and unified to better fit together. The results gathered in the appendix to Chapter 5 were published in [24, 25, 27, 29]. All the theorems of Chapter 1 are classical and well known. The intend of the author was to make the book as self-contained as possible, to save the reader from the need to refer to any external sources while reading. Hence the opening chapter contains a basic introduction to the theory of bilinear forms, valuations and orderings, serving as a handy reference for the following chapters. The presentation of the first chapter is necessarily brief and most of the proofs are omitted.

A short remark on the notational convention. We tend to use letters P and R to denote rings. While the latter is widely used and self-explaining (as the first letter of the word "ring"), the use of P requires some justification. We often need to compare two rings and so R alone is not enough. The letter P is close enough to R so that we can write "P is a subring of R" with the inclusion relation preserving the natural alphabetical order. Secondly, P is the first letter of the Polish word "pierścień" (which means ring), hence we have P for "pierścień" and R for "ring". The rest of notation used in this book is standard and agrees with broadly accepted conventions. For reader's convenience, we include the list of commonly used symbols on page 89.

Commonly used symbols

$\rightarrowtail$	— injection/monomorphism
$\xrightarrow{\sim}$	— bijection/isomorphism
$\twoheadrightarrow$	— surjection/epimorphism
$\langle u_1, \dots, u_k \rangle$	— diagonal form (see p. 13)
$\langle\langle u_1, \dots, u_k \rangle\rangle$	— Pfister form (see p. 13)
$\left(\frac{a,b}{K}\right)$	— quaternion algebra over K
Θ_V	— null vector of a vector space V
$\mathbb{A}^n K$	— n-dimensional affine K-space
ann	— annihilator
Br	— Brauer group/functor
$\mathfrak{c}$	— conductor of a ring extension
$\chi_{(\mathfrak{p},\mathfrak{q})}$	— interval function (see p. 26)
$D_K(\xi)$	— set of elements represented by ξ in K
det	— determinant
disc	— discriminant (see p. 13)
∂	— second residue homomorphism (see p. 20)
$\mathbb{E}$	— see p. 23
γ, γ_K	— real algebraic curve
$G_P(\xi)$	— group of similarity factors of ξ (see p. 37)
$H(a)$	— element of a subbasis of Harrison topology (see p. 20)
$\mathcal{H}$	— real holomorphy ring (see p. 76)
$\mathrm{Hom}_P(M, N)$	— module of homomorphism of P-modules M, N
im	— image of a morphism
ker	— kernel of a morphism
$\dot{K}/\dot{K}^2$	— square-class group of K

$\mathbb{N}$	— positive integers
$\mathbb{N}_0$	— non-negative integers
Nil	— nilradical (see p. 19)
$\mathcal{O}, \mathcal{O}_\mathfrak{p}, (\mathcal{O}_\mathfrak{p}, \mathfrak{p})$	— valuation ring
$\Omega, \Omega(K)$	— set of valuation rings if K
$\mathrm{ord}_\mathfrak{p}$	— discrete valuation with a valuation ring $(\mathcal{O}_\mathfrak{p}, \mathfrak{p})$ (see p. 18)
$\dot{P}$	— set of non-zero-divisors of P
Pic	— Picard group/functor
$\mathbb{P}^n K$	— n-dimensional projective K-space
qf	— field of fractions
$\mathbb{Q}$	— field of rationals
Rad	— Jacobson's radical
$\mathbb{R}$	— field of reals
sgn_β	— signature associated with an ordering β
$\mathrm{sgn}_\mathfrak{p}$	— sign at a point $\mathfrak{p}$
Sgn	— total signature
$\sum \dot{P}^2$	— set of sums of squares in P
Spec	— spectrum of a ring
$S^\perp$	— orthogonal completion of S (see p. 12)
s_*	— transfer (see p. 34)
Tor	— set of torsion elements (see p. 19)
UP	— group of units of P
$u(K)$	— u-invariant of K (see p. 74)
WP	— Witt ring/group of P (see p. 15)
$\Xi(\xi, \mathscr{B})$	— matrix of a form ξ in a basis $\mathscr{B}$
$\mathbb{Z}$	— ring of integers

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Morfizmy pierścieni Witta

Streszczenie

Rozprawa *Witt morphisms* omawia właściwości funktora Witta na kategorii pierścieni przemiennych z jedynką. Książka składa się z pięciu rozdziałów, z których pierwszy ma charakter wprowadzający do tematyki funktora Witta. W rozdziale tym są zdefiniowane kluczowe pojęcia niezbędne do rozumienia dalszych części pracy i przywołane standardowe wyniki używane w kolejnych rozdziałach. Główną część pracy stanowią rozdziały 2–5. Rozdział drugi omawia problematykę zachowania funktora Witta na rozszerzeniach unitarnych (a w szczególności na kwadratowych rozszerzeniach unitarnych) pierścieni lokalnych. Rozdział ten zawiera między innymi uogólnienie techniki transferowej Scharlau'a na przypadek rozszerzeń niewolnych. Rozdział trzeci wykorzystuje wyniki rozdziału poprzedniego do badania zachowania funktora Witta normalizacji dziedzin wymiaru jeden. W rozdziale tym w szczególności poruszana jest kwestia (nie)injektywności funktora Witta normalizacji. Rozdział czwarty poświęcony jest tematyce rozszczepialności ciągu dokładnego Knebuscha–Milnora dla pierścieni geometrycznych. Ostatni, piąty, rozdział rozprawy dotyczy równoważności Witta rzeczywistych ciał i pierścieni, czyli istnienia izomorfizmu między pierścieniami Witta dwóch struktur algebraicznych.

Przemysław Koprowski

Wittmorphismen

Zusammenfassung

Das Buch *Witt morphisms* befasst sich mit den Eigenschaften des Wittfunktors in der Kategorie der kommutativen Ringe mit Eins. Das Buch hat fünf Kapitel. Das erste gibt eine Einfuehrung in die Terminologie und die klassischen Resultate, die für die weiteren Kapitel notwendig sind. Die Hauptresultate des Buches sind in den Kapiteln 2–5 enthalten. Das zweite Kapitel diskutiert das Verhalten des Wittfunktors unter unitären Erweiterungen (inbesondere unter quadratischen unitären Erweiterungen) lokaler Ringe. Neben anderen Themen enthaelt dieses Kapitel eine Verallgemeinerung von Scharlaus Transferprinzip für nicht-freie Erweiterungen. Gegenstand des dritten Kapitels ist das Verhalten des Wittfunktors unter Normalisierung. Eines der Hauptthemen ist die Nicht-Injektivität des Wittfunktors unter Normalisierung. Das vierte Kapitel beschäftigt sich mit dem Problem der Zerfällung der Knebusch–Milnor exakten Folge für geometrische Ringe. Das letzte, fünfte Kapitel behandelt die Theorie der Witt-Äquivalenz formal reeller Ringe und Körper.

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